#### <span id="page-0-0"></span>Timing Channels and Shift-Correcting Codes

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Network Coding and Designs Dubrovnik, Croatia

April 4, 2016

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• Zero-error code is a code with the probability of error equal to zero (under optimal decoding)

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	- $\iff$  No two codewords can produce the same output
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C. E. Shannon, "The Zero Error Capacity of a Noisy Channel," IRE Trans. Inf. Theory 2 (3), 1956.

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Example: The BEC



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Example: The BEC



Observation: No matter which codeword is sent, the sequence  $EE \cdots E$  can be received with positive probability

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Example: The BEC



- Observation: No matter which codeword is sent, the sequence  $EE \cdots E$  can be received with positive probability
	- Every two codewords are confusable  $\Longrightarrow$  the zero-error capacity of the BEC is equal to zero

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Another Example:



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**•** Another Example:



confusable, we can use only them and communicate error-free • Now, since the symbols 0 and 2 for example are not

 $\mathbf{E} = \mathbf{A} \oplus \mathbf{A} + \mathbf{A$ 

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1 We can transmit one bit per channel use in this way

We can do better by looking at sequences of length two:

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- 00, 12, 24, 31, and 43 are non-confusable
- The rate of this code is  $\frac{1}{2} \log 5 = \log \sqrt{5}$
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	- L. Lovasz, "On the Shannon Capacity of a Graph," IEEE Trans. Inf. Theory 25 (1), 1979. (IT paper award)

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• Units of transmission: packets

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- Units of transmission: *packets*
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	- 1) Time is slotted, meaning that the packets are sent and received in integer time instants;
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	- 3) Every packet is delayed in the channel for a number of slots chosen randomly from the set  $\{0, 1, \ldots, K\}$ ;

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	- 4) The packets are indistinguishable, and hence the information is conveyed via timing only.

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#### The Shift Channel 1 1 0 1 0 1 0 1 0 1

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- **•** The basic model:  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$  0  $1$ 
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	- $\ket{4}$  The packets are indistinguishable, and hence the information is conveyed via timing only.



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• This model is equivalent to a discrete-time queue with bounded residence times

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- $\bullet$  If the duration of transmission is *n* slots, the transmitted sequence of packets can be identified with a binary sequence from  $\{0,1\}$ <sup>n</sup> 1 2 3 1 2 3



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## The Shift Channel: Comments

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	- V. Anantharam and S. Verdu, "Bits Through Queues," IEEE Trans. Inf. Theory 42 (1), 1996. (IT paper award)  $\sum_{i=1}^{n} a_i$   $\sum_{i=1}^{n}$
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### Optimal zero-error codes

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• Observation: The Shift Channel does not affect the Hamming weight of the transmitted codeword

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- Sequences of length  $n$  and weight  $W$  can be represented as W-tuples of integers  $(p_1, \ldots, p_w)$ , where  $p_i$  is the position of the i'th 1 in the sequence

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 $\bullet$  10010  $\longleftrightarrow$  (1,4)

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- $\bullet$  10010  $\longleftrightarrow$  (1,4)
- Example:  $n = 9$ ,  $W = 2$ ,  $K = 1$

 $(1,2)$ ∩  $(1,3)$   $(2,3)$ ∩ ∩ (1,4) (2,4) (3,4) ∩ ∩ ◯ (2,5) (3,5) (4,5)  $(1,5)$ ◯ ∩ ∩ ◯  $(1,6)$   $(2,6)$   $(3,6)$   $(4,6)$   $(5,6)$ ∩ ∩ ∩ Ω  $(1,7)$ (2,7) (3,7) (4,7) (5,7) (6,7) Ω ∩ ∩ ∩ ∩ Ω (2,8) (3,8) (4,8) (5,8) (6,8) (7,8) (1,8) ◯ ∩ ∩ ∩ ∩ Ω (2,9) (3,9) (4,9) (5,9) (6,9) (7,9) (8,9) (1,9) (2,9) O ∩ ∩

```
(0,0)
(0,1) (1,1)∩
       ∩
(0,2) (1,2) (2,2)∩
       ∩
             ( )
(0,3) (1,3) (2,3) (3,3)◯
       ∩
                   Ω
             ∩
(1,4) (2,4) (3,4) (4,4)
(0,4)
 ∩
       ∩
             ∩
                   ∩
                         Ω
(0,5) (1,5) (2,5) (3,5) (4,5) (5,5)Ω
       ◯
             ∩
                   ∩
                         ∩
                               ∩
(1,6) (2,6) (3,6) (4,6) (5,6) (6,6)
(0,6)
 ◯
       ∩
             ◯
                   ∩
                         ∩
                                     Ω
(1,7)
(2,7) (3,7) (4,7) (5,7) (6,7) (7,7)
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(0,0)
(0,1) (1,1)∩
       ∩
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       ∩
             ( )
(0,3) (1,3) (2,3) (3,3)◯
       ∩
                   Ω
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(1,4) (2,4) (3,4) (4,4)
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 ∩
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(1,6) (2,6) (3,6) (4,6) (5,6) (6,6)
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## Optimal zero-error codes: Constant-weight case

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### Optimal zero-error codes: Constant-weight case

 $\bullet$  Optimal code of length *n* and weight *W* is given by

$$
\mathcal{C}(n, W) = \left\{ \mathbf{x} \in \Delta_{n-W}^W \ : \ \mathbf{x} = \mathbf{0} \ \ (\text{mod} \ \ K+1) \right\}
$$

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$$

• The size of the optimal constant-weight code is therefore

$$
M(n, W) = \binom{W + \lfloor \frac{n-W}{K+1} \rfloor}{W}
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 $M(n) = \sum_{W=0}^{n} M(n, W)$ 

# Zero-error capacity

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Since the constructed codes are optimal, the zero-error capacity is equal to

$$
C_0=\lim_{n\to\infty}\frac{1}{n}\log M(n)
$$

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$$

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#### Theorem

The zero-error capacity of the Shift Channel with parameter K is equal to log r, where r is the unique positive real root of the polynomial  $x^{K+1} - x^{K} - 1$ .

### Zero-error capacity

Proof:

 $\bullet$   $M(n)$  can also be described recursively

$$
M(n) = M(n-1) + M(n-K-1)
$$

with  $M(n) = n + 1$  for  $n \leq K$ 

• This implies that

$$
M(n) = \sum_{k=0}^{K} a_k r_k^n
$$

where  $r_k$  are the roots of the polynomial  $x^{K+1} - x^{K} - 1$ , and  $a_k$  are complex constants

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Therefore,  $M(n) \sim ar^n$ , where r is the largest of these roots (which is the unique positive real root)

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• The constant-weight zero-error capacity-

The constant-weight zero-error capacity—the largest rate attainable asymptotically with the requirement that the fraction of the slots used for transmitting packets equals  $\omega$ :

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The constant-weight zero-error capacity—the largest rate attainable asymptotically with the requirement that the fraction of the slots used for transmitting packets equals  $\omega$ :

$$
C_0(\omega) = \lim_{n \to \infty} \frac{1}{n} \log M(n, \omega n) = \frac{\omega K + 1}{K + 1} \mathcal{H}\left(\frac{\omega (K + 1)}{\omega K + 1}\right)
$$

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$$

• The zero-error capacity can be achieved with constant-weight codes, so

$$
\mathit{C}_0 = \max_{\omega \in [0,1]} C_0(\omega) = \frac{\omega^*K+1}{K+1}\mathcal{H}\left(\frac{\omega^*(K+1)}{\omega^*K+1}\right)
$$

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• More precise asymptotics:

$$
\frac{1}{n}\log M(n,\omega^*n)=C_0-\frac{1}{2n}\log n+\mathcal{O}\left(\frac{1}{n}\right).
$$

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- Note: Even though the capacity can be achieved with constant-weight codes, their performance is worse at finite blocklengths
	- This is quantified by the second-order term  $-\frac{1}{2n}$  log n

# **Generalizations**

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### **Generalizations**

 $\bullet$  P types of packets; the delay of each packet is at most  $K$ , as before, and the packets cannot be reordered (queue with a FIFO service procedure)

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• Now the packets themselves also carry information

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- Main observation: We can first design the "timing" code  $(P = 1)$ , and then assign to every such codeword of weight W all possible sequences of packets ( $P^{W}$  of them)

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4 D > 4 P + 4 B + 4 B + B + 9 Q O

•  $P = 2$ : 10010  $\longrightarrow$  A00A0, A00B0, B00A0, B00B0

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- This construction is optimal
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 $\bullet$  P types of packets; additional noise modeled as a memoryless channel "acting" on the packets and having zero-error capacity  $C'_0$ 

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- $\bullet$  P types of packets; additional noise modeled as a memoryless channel "acting" on the packets and having zero-error capacity  $C'_0$ 
	- Replace  $P$  with  $2^{C'_0}$
- Allowed shifts from  $\{-K_1, \ldots, 0, \ldots, K_2\}$ 
	- Equivalent to shifts belonging to  $\{0, \ldots, K_1 + K_2\}$
- Continuous-time channel with emissions separated by at least  $\tau$  seconds, and with the maximum delay of T seconds
	- The capacity equals  $\frac{1}{\tau}$  log r, where r is the unique positive root of the polynomial  $x^{T/\tau} - x^{T/\tau-1} - 1$

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### Zero-error-detecting codes

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	- No codeword can produce another *codeword* at the output
- Zero-error-detection capacity of a channel is the largest rate achievable (asymptotically) with zero-error-detecting codes

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(0,0)  $\circ$  $(0,1)$   $(1,1)$ Ω ∩ (0,2) (1,2) (2,2) Ω ∩  $(0,3)$   $(1,3)$   $(2,3)$   $(3,3)$ O ∩ ∩ (1,4) (2,4) (3,4) (4,4) (0,4)  $\bigcirc$ ◯ ∩ ∩ ∩ (1,5) (2,5) (3,5) (4,5) (5,5) (0,5) O ∩ ∩ ∩ (0,6) (1,6) (2,6) (3,6) (4,6) (5,6) (6,6) ◯ ∩ ∩ ∩ ∩ O (0,7) $(1,7)$   $(2,7)$   $(3,7)$   $(4,7)$   $(5,7)$   $(6,7)$   $(7,7)$ O О O ∩ О ∩

### Zero-error-detecting codes: Example  $(K = 2)$

# Zero-error-detecting codes: Example  $\left(K=2\right)^{-1}$



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• The shifts are now assumed to be  $\in \{-K_1, \ldots, 0, \ldots, K_2\}$ 

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• Suppose also w.l.o.g. that  $K_1 \leq K_2$ 

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$$
\mathcal{D}^{(a)}(n, W) = \left\{ \mathbf{x} \in \Delta_{n-W}^W : \mathbf{x} = \mathbf{0} \text{ (mod } K_1 + 1), \newline \sum_{i=1}^W x_i = a \text{ (mod } WK_2 + 1) \right\}.
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• Example:  $n = 9$ ,  $W = 2$ ,  $K_1 = 0$ ,  $K_2 = 2$ 

#### Zero-error-detecting codes: Construction  $(\mathcal{K}_1=0,\mathcal{K}_2=2)$  $\overline{2}$



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**Claim:** The code  $\mathcal{D}^{(a)}(n, W)$  is zero-error-detecting

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Proof:

- Let y be the received sequence
- If  $\sum_{i=1}^{W} y_i \neq a$  (mod  $\mathit{WK}_2 + 1$ ), the receiver detects an error
- Suppose then that  $\sum_{i=1}^{W} y_i = a \pmod{WK_2+1}$
- This means that the sum of the coordinates has been changed in the channel by a multiple of  $WK_2 + 1$
- Since  $-K_1 \le y_i x_i \le K_2$  in our model, we have  $-WK_1 \leq \sum_{i=1}^{W}(y_i - x_i) \leq WK_2$ , so the sum could not have been changed for a nonzero multiple of  $WK_2 + 1$
- Therefore, the sum wasn't changed at all and, if there were any shifts in channel, some of them must have been shifts to the right and some of them to the left
- Suppose that the *i*'th particle was shifted to the left,  $y_i < x_i$
- Then, since  $x_i$  is a multiple of  $K_1 + 1$ , and  $-K_1 \le y_i x_i < 0$ ,  $y_i$  cannot be a multiple of  $K_1 + 1$ , and so **y** is not a codeword

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C(n, W) = \bigcup_{a=0}^{W K_2} \mathcal{D}^{(a)}(n, W)
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- What is the rate of the codes  $\mathcal{D}^{(a)}(n,W)$ ?
- Note that  $\mathcal{C}(n, W) = \left( \int_{0}^{Wn_2} \mathcal{D}^{(a)}(n, W) \right)$  $WK<sub>2</sub>$  $a=0$
- This implies that, for at least one a

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\left|\mathcal{D}^{(a)}(n,W)\right| \geq \frac{\left|\mathcal{C}(n,W)\right|}{W\mathsf{K}_2+1}
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 $a=0$ 

Asymptotically, as  $n\to\infty$  and  $W\sim \omega n$ , the codes  $\mathcal{D}^{(a)}(n,W)$ have the same rate as the codes  $C(n, W)$  designed for the smaller of the two parameters  $K_1$ ,  $K_2$ 

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• One cannot do better than this (asymptotically):

Every code detecting shifts from  $\{-K_1, \ldots, 0, \ldots, K_2\}$  is a code correcting shifts from  $\{-K_1, \ldots, 0\}$ 

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#### Theorem

The zero-error-detection capacity of the Shift Channel with parameters  $K_1$ ,  $K_2$ , is equal to log r, where r is the unique positive real root of the polynomial  $x^{\min\{K_1,K_2\}+1} - x^{\min\{K_1,K_2\}} - 1$ .

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DTQP—Discrete-Time Queue with bounded Processing times

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- **o** The model:
	- 1) Time is slotted, meaning that the packets are sent and received in integer time instants;

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- The received sequence can be as long as  $(K + 1)n$ (longer than the input for a multiplicative constant!)
- We have to incorporate this fact in the definition of the code rate:

$$
\frac{1}{L_{\text{av}}(n)}\log M(n)
$$

where  $L_{av}(n)$  is the average output length (average over all codewords and channel statistics)

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• Note: We can again focus on the constant-weight case

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• Example:  $n = 9$ ,  $W = 2$ ,  $K = 1$ 

```
(0,0)
(0,1) (1,1)∩
       ∩
(0,2) (1,2) (2,2)∩
       ∩
             ( )
(0,3) (1,3) (2,3) (3,3)◯
       ∩
                   Ω
             ∩
(1,4) (2,4) (3,4) (4,4)
(0,4)
 ∩
       ∩
             ∩
                   ∩
                         Ω
(0,5) (1,5) (2,5) (3,5) (4,5) (5,5)Ω
       ◯
             ∩
                   ∩
                         ∩
                               ∩
(1,6) (2,6) (3,6) (4,6) (5,6) (6,6)
(0,6)
 ◯
       ∩
             ◯
                   ∩
                         ∩
                                     Ω
(1,7)
(2,7) (3,7) (4,7) (5,7) (6,7) (7,7)
(0,7) (1,7)
 O
                   ∩
       ∩
             ∩
                         ∩
```


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### Optimal zero-error codes

Optimal solution:

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- Optimal solution:
	- Forbid the sequences whose 1's are too close to affect each other (i.e., with less than  $K$  zeros in between)

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- The size of the resulting codes is

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M^{\mathbb{Q}}(n, W) = \begin{pmatrix} \frac{n+K}{K+1} \\ W \end{pmatrix}
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• The capacity:

$$
\mathcal{C}_0^{\text{Q}} = \frac{1}{\mathcal{K}+1}
$$

### **Generalizations**

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 $\bullet$  P types of packets, no reordering (FIFO service)

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 $\bullet$  P types of packets, no reordering (FIFO service)

#### Theorem

Zero-error capacity of the DTQP with P types of packets is

$$
C_0^{\text{Q}} = \max \left\{ \frac{\log (P+1)}{K+1} ~,~ \frac{\log P}{\bar{\kappa}+1} \right\}.
$$

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$$

• Even though we are analyzing the zero-error case, the capacity depends on the channel statistics through  $\bar{\kappa} = \sum_{k=0}^K k \cdot p(k)$ 

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# And Finally...

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# Thank you for your attention!

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