Concatenation of Rank Metric Codes with Convolutional Codes (for Video Streaming)

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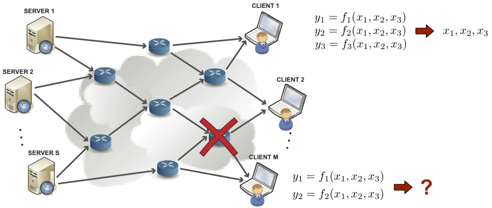
Overview

Linear Network Coding



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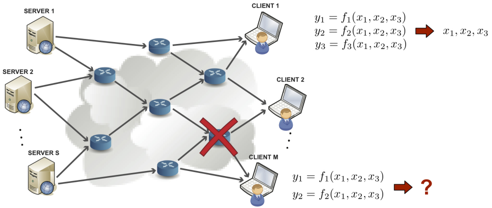

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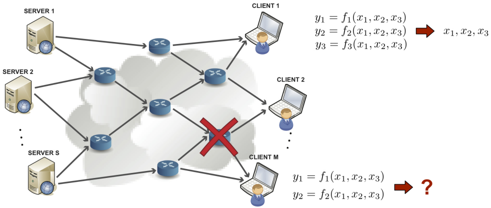

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- These packets propagate through the network. Each node creates a random -linear combination of the packets it has available and transmits this random combination.
- Finally, the receiver collects such randomly generated packets and tries to infer the set of packets injected into the network

Rank metric codes are used in Network Coding

- Rank metric codes are matrix codes $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$, armed with the rank distance

$$d_{\mathrm{rank}}(X,Y) = rank(X - Y), \text{ where } X, Y \in \mathbb{F}_q^{n\times m}.$$

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- The code that achieves this bound is called Maximum Rank Distance (MRD). Gabidulin codes are a well-known class of MRD codes.

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- One standard way to impose correlation of codewords over time is by means of convolution codes.
- We propose to use a concatenated code derived by combining a rank metric code (as inner code) and a convolutional code (as outer code)
- We show how this scheme add complex dependencies to data streams in a quite simple way

Block codes vs convolutional codes

$$\ldots u_2,\ u_1,\ u_0 \xrightarrow{G} \ldots v_2 = u_2 G,\ v_1 = u_1 G,\ v_0 = u_0 G$$

represented in a polynomial fashion

$$\cdots + u_2 D^2 + u_1 D + u_0 \xrightarrow{G} \cdots + \underbrace{u_2 G}_{v_2} D^2 + \underbrace{u_1 G}_{v_1} D + \underbrace{u_0 G}_{v_0}$$

substitute G by $G(D) = G_0 + G_1 D + \cdots + G_s D^s$?

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Block codes: $\mathcal{C} = \{uG\} = \mathrm{Im}_{\mathbb{F}} G \sim \{u(D)G\} = \mathrm{Im}_{\mathbb{F}(D)} G$
Convolutional codes: $\mathcal{C} = \{u(D)G(D)\} = \mathrm{Im}_{\mathbb{F}((D))} G(D)$

A convolutional code $\mathcal{C}$ is a $\mathbb{F}((D))$-subspace of $\mathbb{F}^n((D))$.

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A matrix $G(D)$ whose rows form a basis for $\mathcal{C}$ is called an encoder.
If $\mathcal{C}$ has rank k then we say that $\mathcal{C}$ has rate k/n.

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Remark

One can also consider the ring of polynomials $\mathbb{F}[D]$ instead of Laurent series $\mathbb{F}((D))$ and define $\mathcal{C}$ as a $\mathbb{F}[D]$-module of $\mathbb{F}^n[D]$.

Historical Remarks

- Convolutional codes were introduced by Elias (1955)
- Became widespread in practice with the Viterbi decoding. Widely implemented codes in (wireless) communications. The field is typically $\mathbb{F}_2$ and the rate and degree are small so that the Viterbi decoding algorithm is efficient.
- Renewed interest for convolutional codes over large alphabets trying to fully exploit the potential of convolutional codes.
- Decoding over the erasure channel is easy (Rosenthal et al. 2012).

MDS convolutional codes over $\mathbb{F}$

The Hamming weight of a polynomial vector

$$v(D) = \sum_{i \in \mathbb{N}} v_i D^i = v_0 + v_1 D + v_2 D^2 + \cdots + v_\nu D^\nu \in \mathbb{F}[D]^n,$$

defined as $\mathrm{wt}(v(D)) = \sum_{i=0}^{\nu} \mathrm{wt}(v_i)$.

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- This bound can be achieve if $|\mathbb{F}| > n$

Theorem

Rosenthal and Smarandache (1999) showed that the free distance of convolutional code of rate k/n and degree δ must be upper bounded by

$$d_{\mathrm{free}}(\mathcal{C}) \leq (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1. \tag{1}$$

A code achieving (1) is called Maximum Distance Separable (MDS).

Definition

Another important distance measure for a convolutional code is the jth column distance $d_j^c(\mathcal{C})$, (introduced by Costello), given by

$$d_H^j(\mathcal{C}) \quad = \quad \min\left\{\mathrm{wt}(v_{[0,j]}(D)) \;\mid\; v(D) \in \mathcal{C} \;\text{ and }\; v_0 \neq 0\right\}$$

where $v_{[0,j]}(D) = v_0 + v_1 D + \ldots + v_j D^j$ represents the j-th truncation of the codeword $v(D) \in \mathcal{C}$.

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The column distances satisfy

$$d_H^0 \leq d_H^1 \leq \cdots \leq \lim_{j \to \infty} d_H^j(\mathcal{C}) = d_{\mathrm{free}}(\mathcal{C}) \leq (n-k) \left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1 \right) + \delta + 1.$$

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The construction of MDP boils down to the construction of Superregular matrices (difficult over *small* fields).

Performance over the erasure channel

Theorem (Rosenthal et al. 2012)

Let $\mathcal{C}$ be convolutional code and $d_H^{j_0}$ its $j = j_0$ -th column distance.

If in any sliding window of length $(j_0 + 1)n$ at most $d_H^{j_0} - 1$ erasures occur then we can recover completely the transmitted sequence.

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Remark

The best scenario happens when the convolutional code is MDP.

Example

$$\cdots vv|\overbrace{\star\star\cdots\star\star}^{60}\overbrace{vvv\cdots vv}^{80}\overbrace{\star\star\cdot\star\star}^{60}vv|vv\cdots$$

A $[202, 101]$ MDS block code can correct 101 erasures in a window of 202 symbols (recovering rate $\frac{101}{202}$): $\Rightarrow$ cannot correct this window.

A $(2, 1, 50)$ MDP convolutional code has also 50% error capability. $(L+1)n = 101 \times 2 = 202$. Take a window of 120 symbols, correct and continue until you correct the whole window.

We have flexibility in choosing the size and position of the sliding window.

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$u(D) = u_0 + u_1 D + u_2 D^2 + \cdots \in \mathbb{F}_{q^{mn}}[D]^k$ the information vector.

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We divide

$$v_i = (v_i^0, v_i^1, \ldots, v_i^{K-1})$$

We identify $v_i^j \in \mathbb{F}_{q^{mn}}$ with a vector $V_i^j \in \mathbb{F}_{q^m}^n$ (for a given basis of $\mathbb{F}_{q^m}^n$) and write

$$V_i = (V_i^0, V_i^1, \ldots, V_i^{K-1})$$

and therefore

$$V(D) = V_0 + V_1 D + V_2 D^2 + \cdots \in \mathbb{F}_{q^m}^n[D]^K.$$

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Finally, the **codewords** $X(D)$ of $\mathcal{C}$ are obtained through the matrix $G_I \in \mathbb{F}_{q^m}^{n \times N}$ in the following way:

$$X_i^j = V_i^j G_I,$$

$$X_i = (X_i^0, X_i^1, \ldots, X_i^{K-1})$$

and

$$X(D) = X_0 + X_1 D + X_2 D^2 + \ldots \in \mathcal{C} \subset \mathbb{F}_q^{m \times n}[D]^N.$$

Distance notions

Definition

The *sum rank distance* of $\mathcal{C}$ is defined as

$$d_{SR}(\mathcal{C}) = \min_{0 \neq X(D) \in \mathcal{C}} \text{rank}\,(X(D)) := \min_{0 \neq X(D) \in \mathcal{C}} \sum_{i \geq 0} \text{rank}\,(X_i)$$

where

$$\text{rank}\,(X_i) := \sum_{j=0}^{K-1} \text{rank}\,(X_i^j).$$

And the *column sum rank distance* of $\mathcal{C}$ is defined as

$$d_{SR}^j(\mathcal{C}) = \min_{X(D) \in \mathcal{C} \text{ and } X_0^0 \neq 0} \sum_{i=0}^{j} \text{rank}\,(X_i),$$

We assume throughout the paper that $m > N$.

Theorem

The Sum Rank distance of the concatenated code $\mathcal{C}$ is

$$d_{SR}(\mathcal{C}) = d_{\mathrm{free}}(\mathcal{C}_O) \times d_{\mathrm{rank}}(\mathcal{C}_I).$$

Theorem

The Column Sum Rank distance of $\mathcal{C}$ is

$$d^j_{SR}(\mathcal{C}) = d^j_H(\mathcal{C}_O) \times d_{\mathrm{rank}}(\mathcal{C}_I).$$

where $d^j_H(\mathcal{C}_O)$ is the column distance of $\mathcal{C}$.

Performance

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If in any sliding window of $\mathcal{C}$ of length (j_0+1) at most $d_{SR}^{j_0}-1$ packet losses occur, then we can completely recover the information sequence.

Previous Theorem has some drawbacks:

- Only necessary conditions. There are erasure patterns that do not satisfy the condition of the Theorem but still can be recovered.
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Theorem

Let $d_H^0(\mathcal{C}_o), d_H^1(\mathcal{C}_o), \ldots, d_H^L(\mathcal{C}_o)$ be the distance profile of $\mathcal{C}_o$. Let L_i be the number of packet losses at time instant i. Assume that we have been able to correctly decode up to an instant $t-1$. Then, we can completely decode up to an instant T, $t \leq T$ iff

$$\sum_{i=0}^{s} L_{T-i+t} \leq d_H^s(\mathcal{C}_o) d_{Rank}(\mathcal{C}_I) - 1 \text{ for } s = 0, 1, \ldots, T.$$

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- The last Theorem suggests an algorithm: Exploit the structure of the equations to solve.
- Simulations: Rate of success in recovering for a given probability of losing a packet.
- Little is known about how to construct good convolutional codes:
  - MDP or equivalently: Construction of superregular matrices over small fields.
  - Constructions tailor-made to deal with *burst of erasures* (in both Hamming and Rank metric)

Thanks for the (financial) support!